Dot Product

$(x_0, y_0) \cdot (x_1, y_1) = x_0 x_1 + y_0 y_1$
$= |(x_0, y_0)||x_1, y_1| \cos(\theta)$

$$\cos(\theta) = \frac{(x_0, y_0) \cdot (x_1, y_1)}{|(x_0, y_0)||x_1, y_1|} = \frac{a \cdot b}{|a||b|}$$

$a \cdot b > 0 \to \theta$ is acute,
$a \cdot b < 0 \to \theta$ is obtuse,
$a \cdot b = 0 \to \theta$ is right

$\text{proj}_b\, a = \frac{(a \cdot b)}{|b|^2} b = (a \cdot \frac{b}{|b|}) \frac{b}{|b|}$

Lines

- Standard Eq. of Line in $\mathbb{R}^2$ perpendicular to $\vec{n} = (a, b)$:
$$\vec{n} \cdot (x - x_0, y - y_0) = 0$$
Or equivalently,
$$a(x - x_0) + b(y - y_0) = 0$$
- Vector Eq. of Line in $\mathbb{R}^2$ parallel to $\vec{v} = (a, b)$:
$$(x, y) = (x_0, y_0) + t\vec{v}$$
In $\mathbb{R}^3$ and $\vec{v} = (a, b, c)$ this generalizes to:
$$(x, y, z) = (x_0, y_0, z_0) + t\vec{v}$$
- *Standard parameterization*: 0 ≤ t ≤ 1

- The Parametric Eq. for a line can be found from the Vector Eq.
$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tb$$
- The Symmetric Eq. for a line is found by solving for *t*:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Planes

- Standard Eq. of Plane in $\mathbb{R}^3$ perpendicular to $\vec{n} = (a, b, c)$:
$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$$
$$= a(x - x_0) + b(y - y_0) + c(z - z_0 = 0$$
- Vector Eq. of Plane in $\mathbb{R}^3$ parallel to $\vec{u}$ and $\vec{v}$ (where $\vec{u} \times \vec{v} \neq 0$) :
$$(x, y, z) = (x_0, y_0, z_0) + a\vec{u} + b\vec{v}$$
- The Symmetric Eq. (solve for t)
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Cylindrical Coordinates

$$x = rcos\theta, \quad y = rsin\theta, \quad z = z$$
$$dV = dzdA = rdrd\theta dz$$

Spherical Coordinates

$$r = \rho sin\varphi, \quad x = rcos\theta = \rho sin\varphi cos\theta,$$
$$y = rsin\theta = \rho sin\varphi sin\theta, \quad z = \rho cos\varphi$$
$$dV = \rho^2 sin\varphi d\rho d\varphi d\theta$$

$: \rho \geq 0, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$

Total Mass

$$M = \iiint_T \delta(x, y, z) dV$$

Center of Mass

$$dm = \delta dV, \; M = \iiint_T dm$$

$$\bar{x} = \iiint_T x dm \quad \bar{y} = \iiint_T y dm \quad \bar{z} = \iiint_T z dm$$

- $r_{\text{Centroid}} \rightarrow \delta = 1$

Surface Area (in $\mathbb{R}^3$)

Parameterize Surface R using:

$r(u, v) = (x(u, v), y(u, v), z(u, v))$

$$SA = \iint_R dS = \iint_R |r_u \times r_v| du dv$$

If $r(x, y) = (x, y, f(x, y))$, then
$r_u \times r_v = (-f_x, -f_y, 1)$, and so

$$SA = \iint_R dS = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, du dv$$

Cross Product

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$(a, b, c) \times (c, d, e) = \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} =$$

$$\vec{\imath}\begin{vmatrix} b & c \\ e & f \end{vmatrix} - \vec{\jmath}\begin{vmatrix} a & c \\ d & f \end{vmatrix} + \vec{k}\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$|\vec{n} \times \vec{m}| = |\vec{n}||\vec{m}| \sin(\theta)$$

* Direction → right-hand rule*

Line Integral

Parameterize **C** using:

$r(t) = (x(t), y(t)), \qquad t_1 \leq t \leq t_2$

$$\int_C F \cdot dr = \int_{t_1}^{t_2} F(r(t)) \cdot r'(t) dt$$

Partial Derivatives

$f_{xy} = f_{yx}$, if f_{xy} and f_{yx} are continuous → $\vec{\nabla} f(x, y, z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$

Linear Approximation

$$f(\boldsymbol{x}) \approx f(p) + \vec{\nabla} f(p) \cdot (\boldsymbol{x} - p)$$

Or as a linearization:

$$L_f(\boldsymbol{x}; p) = f(p) + \vec{\nabla} f(p) \cdot (\boldsymbol{x} - p) \;\rightarrow\; f(\boldsymbol{x}) \approx L_f(\boldsymbol{x}; p)$$

In other words:

$$\Delta f \approx d_p f(\Delta x) = \vec{\nabla} f(p) \cdot (\nabla x)$$

The tangent plane (set) is:

$$z = L_f(\boldsymbol{x}; p)$$

Tangent Plane to Parametric Eq.

$r(u, v) = (x(u, v), y(u, v), z(u, v))$

show r_u and r_v are linearly independent

$$(x, y, z) = p + a\vec{r_u}(u_0, v_0) + b\vec{r_v}(u_0, v_0)$$

Or, $\vec{n} = r_u \times r_v$

$$\vec{n} \cdot ((x, y, z) - p) = 0$$

Hessian Determinant (for checking concavity)

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{|p} = f_{xx} f_{yy} - {f_{xy}}^2$$

D > 0, $f_{xx} > 0$ → local min.
D > 0, $f_{xx} < 0$ → local max.
D < 0 → saddle
D = 0 → degenerate

Lagrange Multiplier

$\vec{\nabla} f$: gradient of the original function, $\vec{\nabla} g$: gradient of the constraint function.

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

Conservative Vector Field

If $F(x, y, z) = \vec{\nabla} f(x, y, z)$,

F**(x,y,z) → conservative vector field, **f**(x,y,z) → potential function of **F

- $\boldsymbol{curl(F) = curl(\vec{\nabla} f) = 0}$
- Fund. Th of Line Integrals

  If **C** is a curve from point **a** to point **b**

$$\int_C F \cdot dr = f(\boldsymbol{b}) - f(\boldsymbol{a})$$

Green's Theorem

Given a region R bounded by a simple, closed curve ∂R (clockwise), and $\boldsymbol{F}(x, y) = (P(x, y), Q(x, y))$, then

$$\int_{\partial R} F \cdot dr = \iint_R curl(F) dA$$

Flux Through a Surface

Flux Integral through (outward) a region M: Parametrize **M** using:

$r(u, v) = (x(u, v), y(u, v), z(u, v))$

n is the unit vector normal to the surface at a point

$\boldsymbol{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$, $\boldsymbol{dS} = |r_u \times r_v| du dv$

$$\iint_M \boldsymbol{F} \cdot \boldsymbol{ndS} = \iint_M F(r(u, v)) \cdot \frac{r_u \times r_v}{|r_u \times r_v|} |r_u \times r_v| du dv$$

$$= \iint_M F(r(u, v)) \cdot (r_u \times r_v) du dv$$

Basic Derivatives

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$
$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2 - 1}}$$
$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}$$
$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}$$

Trigonometric Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\sin(2\theta) = 2\sin(\theta) cos(\theta)$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$
$$= 2\cos^2(\theta) - 1$$
$$= 1 - 2\sin^2(\theta)$$
$$\sin(A + B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\tan^2(\theta) = \sec^2(\theta) - 1$$

Chain Rule for Partial Derivatives

$$\frac{\partial f}{\partial t} = \vec{\nabla} f(\boldsymbol{x}) \cdot \frac{\partial \boldsymbol{x}}{\partial t}$$

Or

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t} + \cdots$$

Directional Derivative

$\vec{u}$ is a unit vector!

$$D_{\vec{u}} f(p) = d_p f(\vec{u}) = \vec{\nabla} f(p) \cdot \vec{u}$$
$$= |\vec{\nabla} f(p)| \cos(\theta)$$

Divergence

$$div(F) = \vec{\nabla} \cdot F = \lim_{A_{(x,y)} \to 0} \frac{1}{|A_{(x,y)}|} \oint_C F \cdot \hat{n} dr$$

Curl

$$curl(F) = \vec{\nabla} \times F = \lim_{A_{(x,y)} \to 0} \frac{1}{|A_{(x,y)}|} \oint_C F \cdot dr$$

The Divergence Theorem

∂R is a ccpr-surface, without boundary, bounding a region R (outwards):

$$\iint_{\partial R} \boldsymbol{F} \cdot \boldsymbol{ndS} = \iiint_R div(F) dV$$

"The Flux through a boundary is the sum of all sources and sinks within the bounded region"

Stokes' Theorem

Given a ccpr-surface M with a boundary ∂M, oriented such that the surface is on the left of the positive direction of the curve, then

$$\int_{\partial M} F \cdot dr = \iint_M curl(F) \cdot ndS$$

Regular

$$\vec{\nabla} f \neq 0$$

Linearly Independent

$$\vec{u} \times \vec{v} \neq 0$$